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ABSTRACT

This book contains the fifth and sixth chapters of the second course of a pilot mathematics sequence for the seventh and eighth grades. The content of the sequence is to serve as a vehicle for the development of relevant computational skills, mathematical reasoning, and geometric perception in three dimensions and is to reflect the application of mathematics to the social and natural sciences. The material is divided into five types of sections: (1) activities; (2) short reading sections; (3) questions; (4) sections for the student with a weaker background; and (5) sections for the strongly motivated student. The material in the fifth and sixth chapters of the second course include: measurement, scientific notation, and variables and functions. (MN)

MATHEMATICS

FOR JUNIOR HIGH SCHOOL

pilot edition *chapters 5 & 6*

second course

Boston University Mathematics Project

CONTENTS

5. PLANET EARTH

SECTION 1 CONTINENTS AND OCEANS

The planet earth has a volume of 1.08×10^{12} km^3. Its total surface area is 5.13×10^8 km^2, of which the land area makes up only about 1.49×10^8 km^2. Most of this land area is distributed among the seven continents. Table 1 lists the approximate areas of these continents.

TABLE 1: The Continents

Name	Area in km^2
Africa	2.99×10^7
Antarctica	1.43×10^7
Asia	4.42×10^7
Australia	7.72×10^6
Europe	9.74×10^6
North America	2.44×10^7
South America	1.77×10^7

How can we compare the areas of these continents to get some idea of the distribution of the earth's land areas? One way to compare quantities is to rank them in order of size. To be able to do this for the areas of the continents, we need to be able to decide which of the numbers in exponential notation is largest. There are two possibilities: Either the numbers have the same power of ten or they do not. If the powers of ten are the same we compare the coefficients.

For example, which has the larger area, Africa or Antarctica? The area of Africa is 2.99×10^7 km^2, whereas the area of Antarctica is 1.43×10^7 km^2. The powers of ten are the same, and so the larger number is the one with the greater coefficient. Therefore, Africa has a greater land area than Antarctica.

When two numbers have different powers of ten we can rewrite them with equal powers of ten. For example, which continent has a larger area, Europe or North America? The area of Europe is 9.74×10^6 km^2 and the area of North America is 2.44×10^7 km^2. Let us rewrite 2.44×10^7 km^2, the area of North America, as 24.4×10^6 km^2. We can then see that North America has a greater area than Europe.

Of course, when numbers are given in standard form there is no need for rewriting. We simply look at the powers of ten. The number with the greater power of ten is the larger of the two. Therefore, we can immediately say that the area of North America is larger than the area of Europe because 10^7 is greater than 10^6.

1. Rearrange Table 1 so that the continents are listed in order from the largest to the smallest.

2. In Table 2 the areas of the oceans are listed.

TABLE 2: The Oceans

Name	Area in km^2
Atlantic	8.69×10^7
Arctic	1.33×10^7
Indian	7.37×10^7
Pacific	1.67×10^8

(a) Which ocean has the largest area?

(b) Which ocean has the smallest area?

3. Which is greater?

(a) 3.5×10^5 or 56×10^4

(b) 23×10^6 or 3.1×10^5

(c) 3.7×10^7 or 36×10^6

4. Which of the two numbers is greater?

(a) 1.49×10^{13} cm; 3.6×10^{10} m

(b) 1.49×10^{18} cm^2; 3.6×10^8 km^2

5. To compare two numbers in standard form that have different powers of ten, why is it not necessary to change them to the same power of ten?

Often, just ordering numbers is not enough to tell us what we wish to know. For example, Table 3 presents the world population by continents from 1950 to 1970. If we are describing population growth for any continent during that period of time, we would be interested not only in the fact that the population is greater in 1970 than in 1950 but also by how much it is greater.

TABLE 3: World Population, by Continents, 1950-1970

Continent	1950	1960	1970
Asia	1.38×10^9	1.66×10^9	2.06×10^9
Europe	5.72×10^8	6.39×10^8	7.05×10^8
North America	1.66×10^8	1.99×10^8	2.27×10^8
South America	1.63×10^8	2.14×10^8	2.83×10^8
Africa	2.22×10^8	2.78×10^8	3.44×10^8
Australia	1.3×10^7	1.6×10^7	1.95×10^7

Source: Adapted from U.N. Demographic Yearbook.

To find out by how much one quantity is greater than another, we calculate the difference between them by subtracting one from the other.

To subtract or to add numbers in exponential notation the numbers must have the same power of ten. When this is not the case, the numbers must be rewritten. For example, to calculate the differ-

ence between the population of Asia and the population of Europe in 1970 we would proceed as follows:

$$(2.06 \times 10^9) - (7.05 \times 10^8) = (2.06 \times 10^9) - (0.705 \times 10^9)$$

$$= (2.06 - 0.705) \times 10^9$$

$$= 1.355 \times 10^9$$

Therefore, Asia has approximately 1.36×10^9 or 1.36 billion people more than Europe.

Similarly, to find the combined population of Asia, Africa, and Australia in 1960, we would add.

$$(1.66 \times 10^9) + (2.78 \times 10^8) + (1.6 \times 10^7)$$

$$= (1.66 \times 10^9) + (0.278 \times 10^9) + (0.016 \times 10^9)$$

$$= 1.95 \times 10^9$$

?

6. Table 4 lists figures for U.S. fuel production during 1972 and 1973. Find by how much the production of each fuel has changed from 1972 to 1973.

TABLE 4: U.S. Fuel Production

Fuel	1972	1973
Crude oil (in barrels*)	3.46×10^9	3.36×10^9
Natural gas (in ft^3)	2.25×10^{13}	2.26×10^{13}
Anthracite coal (in short tons**)	7.10×10^6	6.83×10^6

*A barrel of crude oil is equal to 42 gallons.

**A short ton is equal to 2000 pounds.

7. (a) Use Table 3 to find the increase in population from 1950 to 1970 for each of the continents.

(b) Which continent had the greatest increase?

(c) Why do you think Antarctica is not listed?

(d) Find the total world population for 1950, 1960, and 1970.

(e) By how much did the total world population increase from 1950 to 1960? From 1960 to 1970? Which is the greater increase?

8. If you were to answer each of the following questions, decide whether you would compare the quantities involved by ordering, by subtracting, or by both.

(a) Who is the tallest student in your class?

(b) How many more students are in your classroom than in the one next door?

(c) Have you saved enough money to buy the bicycle you saw on sale for $63.98? If not, how much more do you need to save?

(d) Will the carton of books fit underneath the table?

SECTION 2 REPORTING SUMS OF MEASURED QUANTITIES

Suppose that we want to add the following lengths: 3.12 km, 14.1 km, and 8.19 km. The length 14.1 km is known to the nearest 0.1 km. The other two lengths, 3.12 km and 8.19 km, are both known to the nearest 0.01 km. How should the sum be reported? Should we report it to the nearest 0.1 km or to the nearest 0.01 km?

The smallest subdivision to which a measurement is known tells us its <u>precision</u>: the smaller the subdivision, the more precise

the measurement. Therefore 3.12 km and 8.19 km are both more precise than 14.1 km.

The results of adding measured quantities cannot be more precise than the original data used. A sum or a difference should, therefore, be expressed to the same degree of precision as the least precise number used in the calculation.

Adding our distances, we find

$$\begin{array}{rl} & 3.12 \text{ km} \\ & 14.1 \text{ km} \\ + & \underline{8.19 \text{ km}} \\ & 25.41 \text{ km} \end{array}$$

Since the least precise measure, 14.1 km, is known to the nearest 0.1 km, the sum is rounded off to the nearest 0.1 km and, therefore, should be reported as 25.4 km.

9. What is the smallest subdivision to which each of the following measurements is given?

(a) 13.24 m

(b) 132.4 km

(c) 0.1324 km

(d) 1324 cm

10. Round the following measurements as indicated.

(a) 1.247 m to the nearest 0.01 m

(b) 1.247 m to the nearest 0.1 m

(c) $0.84\ km^2$ to the nearest $0.1\ km^2$

(d) 1.247 m to the nearest 1 m

11. Do the following additions and express each result to the right precision.

(a) 326 m + 451.3 m + 49 m + 27.45 m

(b) 12.6 km + 120.31 km + 0.073 km + 2.436 km

(c) 3.49 cm + 133 cm + 2.38 cm + 16.6 cm

Which measurement is given more precisely, 139 m or 6.13 km? The measurement 139 m is known to the nearest meter, whereas 6.13 km is known to the nearest 0.01 km or 10 m. And so we know that 139 m is more precise than 6.13 km.

Note, however, that we cannot say that 3.5 hours is more or less precise than 13.52 m because we cannot compare 0.1 hour and 0.01 m.

12. For each of the following pairs of measurements, tell which of the two measurements is <u>less</u> precise and why.

(a) 6 cm; 2.3 cm

(b) 1.38 m; 3.9 m

(c) 2.34 in.; 2.328 in.

(d) $321\ m^3$; $330.5\ m^3$

13. For each of the following pairs of measurements, tell which of the two measurements is <u>more</u> precise and why.

(a) 32 cm; 3.5 m

(b) 138 m; 3.65 km

(c) 4.2 in.; 60 hours

(d) 0.05 cm^2; 17 cm^3

In Table 1 the areas of North America and Europe are reported to different degrees of precision. The area of North America, 2.44×10^7 km^2, is given to the nearest 10^5 km^2. The area of Europe, 9.74×10^6 km^2, is given to the nearest 10^4 km^2. Therefore the area of North America is reported less precisely than the area of Europe.

If we wanted to calculate the difference between the areas of the two continents we would, therefore, report this difference to the nearest 10^5 km^2. This is the degree of precision of the least precise number used in the subtraction.

$$(2.44 \times 10^7 \text{ km}^2) - (9.74 \times 10^6 \text{ km}^2) = 1.47 \times 10^7 \text{ km}^2$$

Similarly, if we wanted to calculate the sum of these two areas, we would report the sum to the nearest 10^5 km^2.

$$(2.44 \times 10^7 \text{ km}^2) + (9.74 \times 10^6 \text{ km}^2) = 3.41 \times 10^7 \text{ km}^2$$

14. What is the smallest subdivision to which each of the following measurements is given?

(a) 2.9×10^7 km

(b) 7.72×10^6 m

(c) 7.723×10^7 m^3

(d) 2.06×10^9 people

15. Round these measurements as indicated.

(a) 1.342×10^5 m to the nearest 10^3 m

(b) 1.342×10^5 m to the nearest 10^4 m

(c) 1.08×10^{15} km to the nearest 10^{14} km

(d) 1.08×10^{15} km to the nearest 10^{15} km

16. Which of the following measurements is less precise and why?

(a) 1.38×10^9 people; 3.92×10^8 people

(b) 1.66×10^8 m^3; 1.3×10^7 m^3

(c) 9.3×10^6 miles; 1.6×10^{14} miles

(d) 3.2×10^3 km; 5.1×10^3 m

(e) 3.2×10^3 km; 5.134×10^3 m

17. Refer to the paragraph in Section 1 of this chapter where we compared the population of Asia and Europe in 1970. Which answer, 1.355×10^9 or 1.36×10^9, states the difference to the right precision?

18. Refer to the paragraph in Section 1 where the sum of the populations of Asia, Africa, and Australia was calculated. Was the sum given to the right precision? Justify your conclusion.

19. Restate your answers for Problem 6 so that all calculations are given to the right precision.

20. The area of the 48 United States before 1959 was 7.8×10^6 km^2. During that year, Alaska (area = 1.5×10^6 km^2) and Hawaii (area = 1.7×10^4 km^2) became the 49th and 50th states, respectively.

(a) Find the area of the U.S. including Alaska.

(b) Find the area including all 50 states.

SECTION 3 RATIOS

Often we compare numbers by stating how many times one is larger than the other rather than by how much they differ. For example, an 80-pound dog is 8 times as heavy as a 10-pound cat although they differ by 70 pounds. Or a \$4000 car costs 40 times as much as a \$100 bicycle whereas the difference between them is \$3900.

To find how many times a quantity x is as great as a quantity y, we calculate the ratio of x to y; that is, we divide x by y. Thus to find how many times larger \$15 is than \$3 we divide \$15 by \$3. The ratio $\frac{\$15}{\$3} = 5$ tells us that \$15 is 5 times as large as \$3. However, we might want to find how many times \$3 is as large as \$15. The ratio $\frac{\$3}{\$15}$ or $\frac{1}{5} = 0.2$ tells us that \$3 is 0.2 times as large as \$15.

If two quantities are to be compared by finding their ratio, they must be expressed in the same units. If they are given in different units, one of the quantities has to be converted to the unit of the other quantity. Let us look at an example: How many times is 2400 cm as large as 12 m? By converting 2400 cm into meters, we have 2400 cm = 24 m. The ratio $\frac{24 \text{ m}}{12 \text{ m}} = 2$ tells us that 2400 cm is 2 times as large as 12 m.

21. Find the ratio of

(a) 36 m to 10 m

(b) fifty dollars to thirty dollars

(c) \$0.25 to \$6.25

(d) $\frac{800 \text{ km}}{64 \text{ km}}$

(e) 6 cm to 48 cm

22. Find the ratio of

(a) \$5.00 to 25¢

(b) 6.2 m to 40 cm

(c) 4 m to 4 cm

(d) 36 km^2 to 36 m^2

(e) one quarter to twenty-five dimes

(f) 120 minutes to 2 hours

23. Which is heavier, the dog or the cat, if the ratio of the weight of the dog to the weight of the cat is

(a) greater than 1?

(b) less than 1?

(c) equal to 1?

24. Boeing 707 jets can travel at a speed of 600 mph, and supersonic transports (SST) can go 1400 mph.

(a) Will the ratio of the speed of the SST to the Boeing 707 be greater than, less than, or equal to 1?

(b) Calculate this ratio.

(c) By how much do the two speeds differ?

25. The fastest speed recorded for a human being is 29.89 mph. A cheetah (hunting leopard) is the fastest land animal and can run about 60 mph. A human being runs how many times as fast as a cheetah?

26. Use Table 5 to answer the following questions.

(a) How many times as many people speak Chinese as speak Russian?

(b) Find the ratio of the number of Portuguese-speaking people to the number of Spanish-speaking people.

(c) How many times as many people speak English as speak French?

TABLE 5: Principal Languages of the World

Language	Number of Speakers (in millions)
Arabic	98
Chinese	709
English	345
French	75
German	120
Hindi	181
Japanese	102
Portuguese	97
Russian	188
Spanish	179

27. At birth a blue whale is about 7 meters long and weighs about 2.3 tons. An adult blue whale is approximately 22 meters long and weighs about 136 tons.

(a) What is the ratio of an adult's weight to a newborn's weight?

(b) What is the ratio of an adult's length to a newborn's length?

SECTION 4 AREA AND POPULATION MAPS

Whenever we have facts about regions of the Earth such as continents or countries, we can present these facts dramatically by using maps. For example, the data listed in Table 1 of this chapter can be dramatized by using a map such as the one in Figure 1.

Such a map does not faithfully show the shapes of the continents since it represents each one by a rectangle. The map, however, does vividly illustrate the relative sizes of the continents. For example, the area of the rectangle for Africa is about twice the area of the rectangle for Antarctica, and so we can tell that Africa is twice as large as Antarctica.

The little square in the lower right-hand corner of the map serves as a scale. With it we can calculate the area of each continent. For example, Australia is represented by a rectangle that

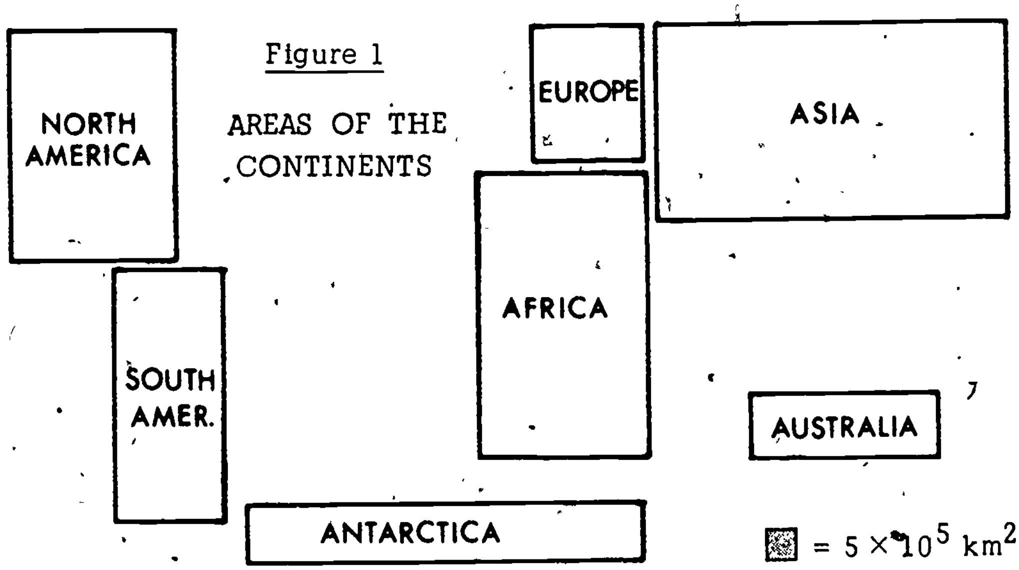


Figure 1

AREAS OF THE CONTINENTS

has an area of about 15 times the area of the little square. The area of Australia is $15 \times (5 \times 10^5)\ \mathrm{km}^2 = 75 \times 10^5\ \mathrm{km}^2$, or about $8 \times 10^6\ \mathrm{km}^2$.

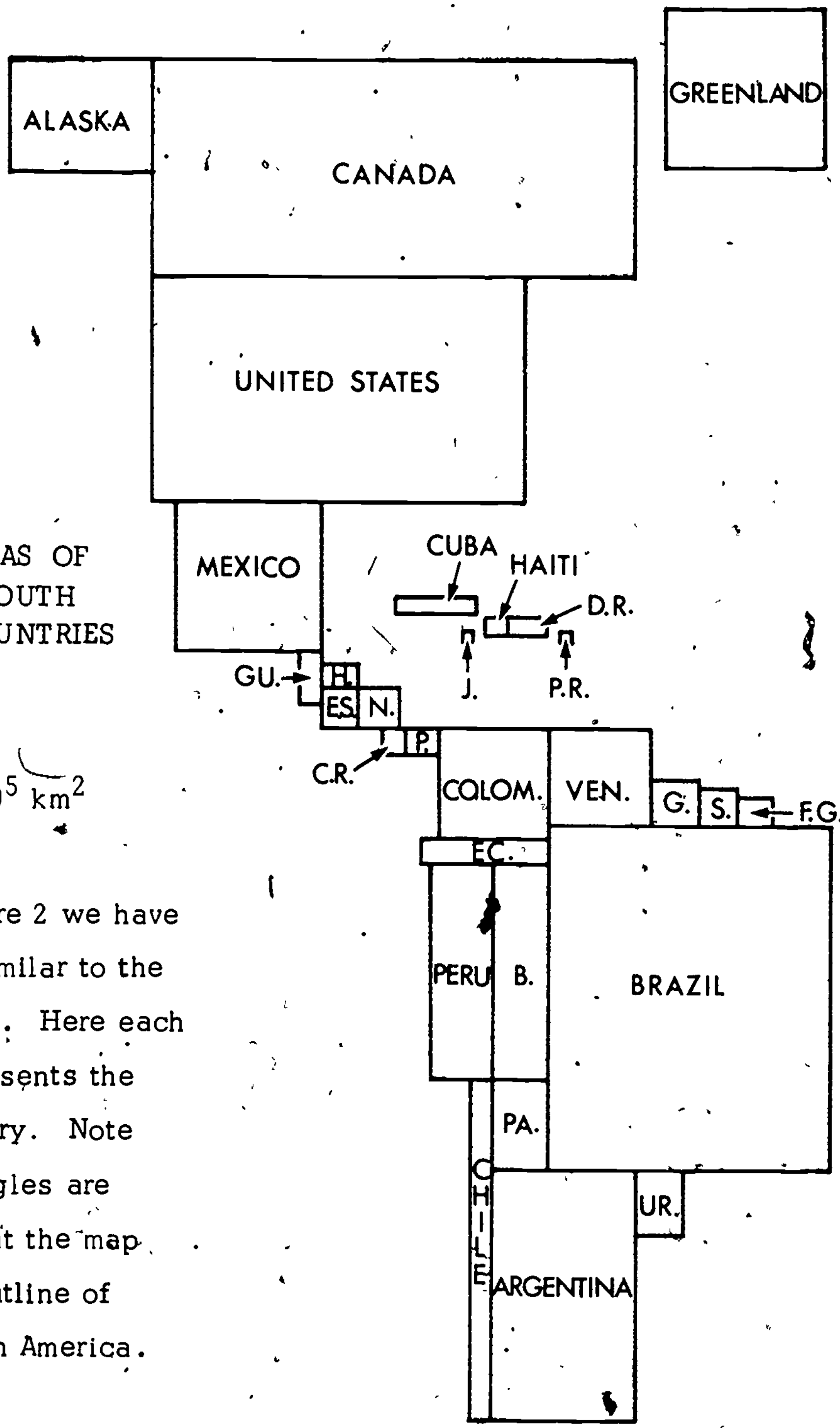


Figure 2 AREAS OF NORTH AND SOUTH AMERICAN COUNTRIES

$= 5 \times 10^5\ \mathrm{km}^2$

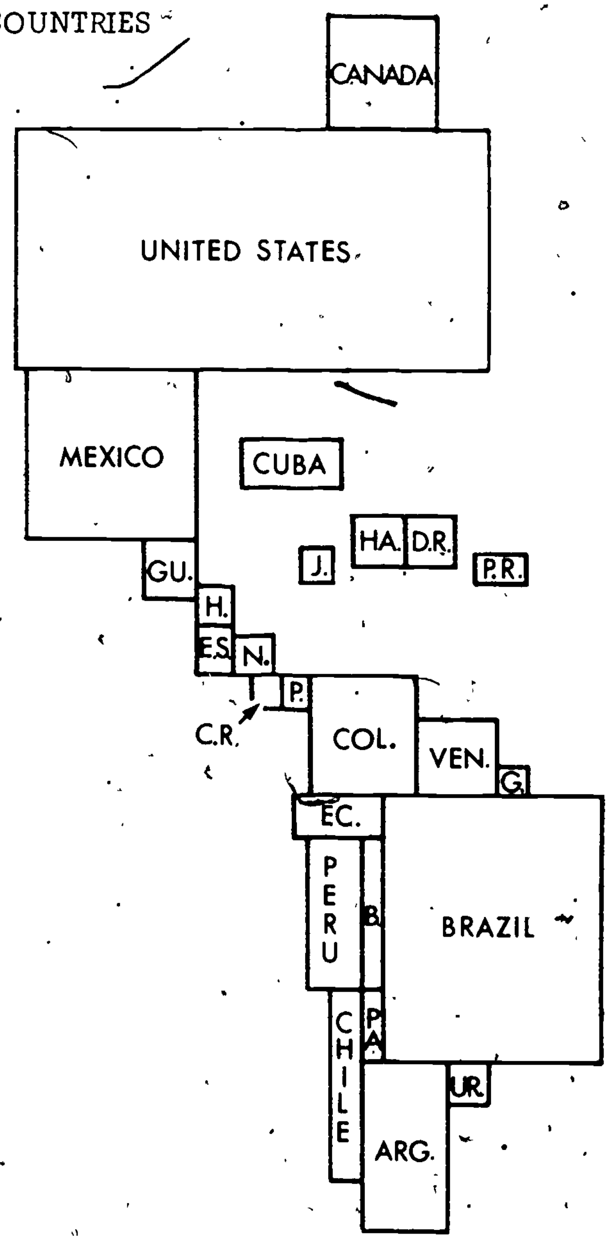
In Figure 2 we have another map similar to the one in Figure 1. Here each rectangle represents the area of a country. Note how the rectangles are arranged so that the map takes on the outline of North and South America.

Area is not the only quantity that can be displayed on such maps. Figure 3 is another map of the countries of North and South America. In this map each rectangle represents the population of each country. Note, for example, how small Canada appears in relation to the United States. Contrast this with the map in Figure 2.

Figure 3 1973 POPULATIONS OF NORTH AND SOUTH AMERICAN COUNTRIES

= 10^7 people

ARG. – Argentina
B. – Bolivia
C.R. – Costa Rica
COL. – Colombia
D.R. – Dominican Republic
E.S. – El Salvador
EC. – Ecuador
G. – Guyana
GU. – Guatemala
H. – Honduras
HA. – Haiti
J. – Jamaica
N. – Nicaragua
P. – Panama
P.R. – Puerto Rico
PA. – Paraguay
UR. – Uruguay
VEN. – Venezuela



28. Which do you think is more crowded — the United States or Canada?

29. Why do you think Greenland is missing from the map in Figure 3?

30. The area of the State of Alaska is represented by a separate rectangle in Figure 2. However, in Figure 3 there is no separate rectangle for the population of Alaska. Why do you think this is so? If you were to draw the rectangle what would you use for the dimensions? (The population of Alaska in 1973 was 3×10^5 people.)

31. The rectangle in Figure 3 for the population of the United States has dimensions of 3.2 cm $\times$ 6.5 cm. Suggest other dimensions that could have been used.

Get together in groups. Use the data in Table 3 of this chapter to make a map of the continents that reflects the population of each continent in 1970. Compare this map with the map of Figure 1.

In an almanac, look up the land area and population of your state and all adjacent states. From this data, make up two maps, one to display the area of each state, and one to display the population.

SECTION 5 NEGATIVE EXPONENTS

32. Multiply:

(a) 1000×10000

(b) 1000000×100

(c) $10^5 \times 10000$

(d) $10^3 \times 10^6$

33. Divide:

(a) $1000000 \div 100$

(b) $\frac{10^5}{10}$

(c) $10^7 \div 10^3$

(d) $\frac{10^8}{10^6}$

The quickest way to multiply two powers of ten is to add their exponents. Similarly, the quickest way to divide two powers of ten is to subtract their exponents.

Examples: $10^4 \times 10^7 = 10^{4+7} = 10^{11}$

$$10^6 \div 10^4 = 10^{6-4} = 10^2$$

However, what do you do with the following division problem?

$$10^2 \div 10^5 = ?$$

If we subtract the exponents, we get

$$10^2 \div 10^5 = 10^{2-5} = 10^{-3}$$

On the other hand, if we do the division by simplifying the fraction $\frac{10^2}{10^5}$, we get

$$\frac{10^2}{10^5} = \frac{10 \times 10}{10 \times 10 \times 10 \times 10 \times 10} = \frac{1}{10^3}$$

And so we can apply the rule for dividing powers of ten to such problems as $10^2 \div 10^5$ by defining 10^{-3} to be equal to $\frac{1}{10^3}$. In fact, this indicates how to give meaning to any negative exponent:

$$\text{By } 10^{-n} \text{ we mean } \frac{1}{10^n}$$

where $\underline{n}$ is any positive integer.

With this definition of negative exponents, we can always use the quick way of dividing two powers of ten, whether the resulting exponent is positive or negative.

Note that 10^{-3} is <u>not</u> a negative number. In fact, 10^{-3} is a number between 0 and 1 since

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1000} = 0.001$$

In general, 10^{-n} is a number between 0 and 1.

34. Express the following as fractions and as decimals.

(a) 10^{-1}

(b) 10^{-2}

(c) 10^{-6}

(d) 10^{-4}

35. Express the following by using negative exponents.

(a) $\frac{1}{10}$

(b) $\frac{1}{10^3}$

(c) $\frac{1}{10 \times 10 \times 10 \times 10}$

(d) $\frac{1}{1000000}$

36. Use negative exponents to express the following decimal numbers.

(a) 0.1

(b) 0.01

(c) 0.0001

(d) 0.00001

37. Divide:

(a) $10^3 \div 10^5$

(b) $10^4 \div 10^5$

(c) $10^4 \div 10^3$

(d) $10^4 \div 10^7$

38. Divide:

(a) $\frac{1000}{1000}$

(b) $\frac{10}{10}$

(c) $\frac{10^2}{10^2}$

(d) $10^5 \div 10^5$

39. How can we define 10^0 in order to give it meaning?

40. Order each group of numbers from smallest to largest.

(a) 10^{-2}, 0, 10^{-1}, 1, 0.5

(b) 10^3, 10^{-1}, 0, 10, 10^0, 10^{-2}, -2

Remember how we change 2.7×10^3 into decimal form. First we use the meaning of 10^3 to tell us that 2.7×10^3 means

$$2.7 \times 10^3 = 2.7 \times 10 \times 10 \times 10$$

Next, we actually perform the multiplication.

A very quick way to multiply by ten three times is to shift the decimal point three places to the right.

$$2.7 \times 10^3 = 2\,7\,0\,0 = 2700$$

Each movement of the decimal point one place to the right corresponds to multiplying the number once by ten.

How do we change 3.1×10^{-2} into decimal form? We know that 10^{-2} means $\frac{1}{10^2}$. Therefore,

$$3.1 \times 10^{-2} = 3.1 \times \frac{1}{10^2} \quad \text{or} \quad 3.1 \div 10^2$$

To divide by 10^2 we can divide by ten twice. Can we perform this division in a quick way by moving the decimal point? Yes, but this time we move the decimal point two places to the left.

$$3.1 \times 10^{-2} = 0\,0\,0\,3\,1 = 0.031$$

and fill in the spaces with zeros. Each movement of the decimal point one place to the left corresponds to dividing by ten once.

In Chapter 3 we learned how to express large numbers in exponential notation by moving the decimal point. For example,

$$360000 = 3.6 \times 10^{\square}$$

What exponent goes in the box? To find out, we start with the coefficient 3.6 and move the decimal a certain number of places so that we get 360000.

$$3\underbrace{60000}$$

We moved the decimal point five places to the right, and so

$$360000 = 3.6 \times 10^{5}$$

Moving the decimal point also works when the original number requires a negative power of ten. For example,

$$0.0583 = 5.83 \times 10^{\square}$$

$$\underbrace{0\,5}\,8\,3$$

In this case we must move the decimal point two places to the left. And so we write

$$0.0583 = 5.83 \times 10^{-2}$$

[?]

41. Write the following numbers in ordinary decimal notation.

(a) 4.3×10^{-2}

(b) 6.67×10^{-3}

(c) 5.16×10^{4}

(d) 1.71×10^{0}

42. Fill in the box with the correct exponent.

(a) $0.034 = 3.4 \times 10^{\square}$

(b) $0.0015 = 1.5 \times 10^{\square}$

(c) $0.780 = 7.8 \times 10^{\square}$

43. Fill in the box with the correct exponent.

(a) $0.225 = 2.25 \times 10^{\square}$

(b) $1.60 = 1.60 \times 10^{\square}$

(c) $3700 = 3.7 \times 10^{\square}$

(d) $0.07 = 7 \times 10^{\square}$

(e) $3.8 = 3.8 \times 10^{\square}$

44. Note that 5×10^3, 5×10^0, and 5×10^{-2} are all positive numbers, regardless of the sign of the exponent. We can also represent negative numbers in exponential notation by making the coefficient negative.

For example, $-5000 = -5 \times 10^3$

and $-100 = -1 \times 10^2$

and $-0.3 = -3 \times 10^{-1}$

Write the following numbers in standard form.

(a) -45

(b) -23,000

(c) -0.089

(d) -0.00016

(e) -1.50

45. Suppose you write a number, N, in standard form. State what the exponent of the power of ten must be if

(a) $1 < N < 10$

(b) $10 < N < 100$

(c) $\frac{1}{10} < N < 1$

(d) $0 < N < 1$

(e) $1 < N$

SECTION 6 ORDERS OF MAGNITUDE

Giant redwood trees are not all the same height – some may be 60 m high and others, perhaps, 110 m. A useful way to describe the general size of redwoods is to say that they are around 10^2 m high.

When a quantity, such as the size of a redwood, is estimated by the power of ten closest to the number, the estimate is called the order of magnitude of the number. For example, human beings, whether newborn babies or basketball players, are of the order of 10^0 m. Flies and bees are of the order of 10^{-2} m.

We determine the order of magnitude of a number by deciding which power of ten is closest to it. For example, the order of magnitude of 2300 is 10^3; that is, 2300 is closer to $10^3 = 1000$ than it is to $10^2 = 100$ or $10^4 = 10,000$. Similarly, the order of magnitude of $8 \times 10^3 = 8000$ is 10^4 because 8×10^3 is closer to 10^4 than to 10^3.

Orders of magnitude are most useful when we wish to compare quantities that can vary over a vast range. Table 6 illustrates this point by giving us a rough idea of the sizes of objects. To compare such widely different numbers by ratio, we often need to compare only their orders of magnitude. For example, let us compare the size of a redwood and the size of bacteria. From Table 6 we see that the tree is $\frac{10^2}{10^{-6}} = 10^8$ times as large as bacteria. Or, in other words, the tree is 8 orders of magnitude larger.

TABLE 6: Sizes of Objects on Earth (in meters)

Distance from Equator to North Pole	10^7	(10,000 km)
	10^6	(1000 km)
	10^5	
Height of Mt. Everest	10^4	
	10^3	(km)
Giant trees: redwoods	10^2	
Whales	10^1	
Elephants; human beings	10^0	(meter)
Small birds and mammals; large insects	10^{-1}	
Small insects; tiny fish	10^{-2}	(cm)
	10^{-3}	
Protozoa; grains of pollen	10^{-4}	
Human blood-cells	10^{-5}	
Bacteria	10^{-6}	(micron)

Source: Adapted from D'Arcy Thompson, On Growth and Form (Cambridge University Press, 1915).

When we say that a number is one order of magnitude greater than another, we mean that it is ten times as great as that number. Similarly, if it is 2 orders of magnitude greater, it is $10^2 = 100$ times as great. For example, 9.28×10^3 is 5 orders of magnitude greater than 2.67×10^{-1} because 10^4 is 10^5 times as large as 10^{-1}. Note that 3×10^2 and 2×10^2 would be considered to have the same order of magnitude. The order of magnitude of both numbers is 10^2.

46. Estimate the following as orders of magnitude.
    (a) The width of this page in cm
    (b) The length of one mile in m
    (c) The width of your hand in m
    (d) The diameter of a pencil in m
    (e) The thickness of this book in m

47. Use Table 6 to answer the following questions. Give your answers in terms of orders of magnitude.
    (a) How many times larger is a whale than a human being?
    (b) How many times larger is a human being than bacteria?
    (c) How many times larger is a whale than bacteria?

48. Table 6 indicates that the height of humans and the height of elephants are of the same order of magnitude. Show that this is true by looking up some actual sizes.

49. Within how many orders of magnitude are mammals that either walk or run?

50. (a) Are the areas of all the continents of the same order of magnitude?

(b) Are the areas of all the oceans of the same order of magnitude?

51. Which would represent the greater change for a given number — increasing the number by one order of magnitude or doubling the number?

52. Since computers can calculate much more rapidly than people can, computers have lowered the cost of doing calculations by 3 orders of magnitude. What would a car cost if its price were to drop by 3 orders of magnitude?

53. Obtain a copy of the Guinness Book of World Records to help you answer the following questions.

How many orders of magnitude

(a) heavier was the largest fish ever caught with a fishing rod than the smallest fish?

(b) faster was the fastest typist in English than the fastest typist in Chinese?

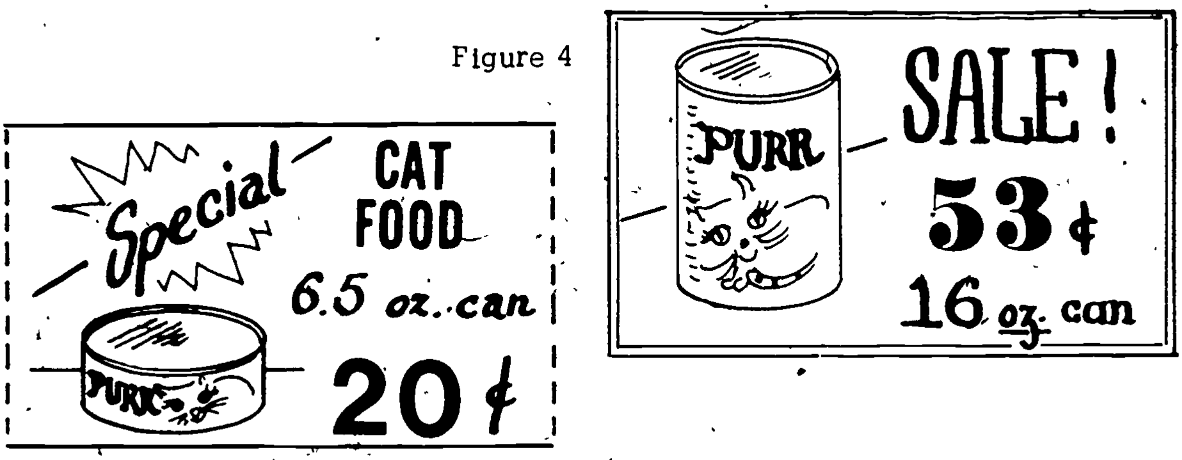
(c) longer was the longest scheduled flight than the shortest one?

SECTION 7 THE MEANING OF "PER"

You often see ads like those of Figure 4. If you were buying food for your cat, which can should you buy? The amount of cat food in each can is different, and so to decide which is a better buy, you should not compare the advertised prices directly. Assuming that the quality of the food is the same in both cans, we should compare the cost of one ounce.

In calculating the cost of one ounce we divide the cost of a can by the number of ounces in the can. The result of the division is the cost of one ounce or the cost per ounce. In general, this quantity is known as the unit price. Since the unit price depends on the unit of weight used, this unit is written explicitly in the calculation and in the result.

Figure 4



In our example the 6.5-ounce can costs 20¢. Therefore the unit price of cat food in that can is $\frac{20¢}{6.5\ \text{oz}} = 3.08¢/\text{oz}$, which we read as "3.08 cents per ounce." In comparison, if you were to buy the large can, you would pay $\frac{53¢}{16\ \text{oz}} = 3.31¢/\text{oz}$. Therefore, in this case the small can is a better buy.

Let us look at another comparison. An ice skater covers 1000 meters in 1.5 minutes. A second ice skater covers 1500 meters in 2.3 minutes. Which skater is faster? Knowing only the skating time or only the distance covered is not enough to provide an answer. The question "how fast" is answered by distance per unit time, which is called speed.

The first skater covered $\frac{1000 \text{ m}}{1.5 \text{ min}} = 667$ m/min ("667 meters per minute"). The second skater, on the other hand, covered $\frac{1500 \text{ m}}{2.3 \text{ min}} = 652$ m/min. The first skater is therefore faster than the second.

54. In many states grocery stores are required by law to state the unit price for all commodities. Why do you think this law was passed?

55. You can buy your favorite brand of orange soda in a 2-liter bottle for 73¢ or in a 0.3-liter can for 15¢.

    (a) What is the unit price for the bottle? For the can?

    (b) Which is the better buy?

56. A 15-ounce can of peaches costs 23 cents, and a 24-ounce can costs 35 cents. Which can is a better buy?

57. In 1975 U.S. government expenditures were about 3×10^{11} dollars. In order to really understand how much money was spent, figure out how many dollars per person that is (U.S. population = 2×10^{8} people).

58. In the reading section the cost of the can in cents was divided by the weight of the cat food in ounces. We could also have divided the weight in ounces by the cost in cents. For example, for the large can we would get $\frac{16 \text{ ounces}}{53 \text{ cents}} = 0.30$ ounces for for each cent, or 0.30 oz/¢.

    (a) Does this quantity mean anything, and if so, what?

    (b) What advantage is there to knowing oz/¢ rather than ¢/oz?

59. Describe some situations that would involve quantities given in the following units.

(a) $\frac{\text{meters}}{\text{second}}$

(b) $\frac{\text{dollars}}{\text{person}}$

(c) $\frac{\text{gallons}}{\text{person}}$

(d) $\frac{\text{pounds}}{\text{meter}}$

60. Why do you think speeds of trains or cars are stated in km/hour and speeds of runners in m/sec?

61. A good runner runs 100 m in 12 seconds. What is the runner's speed in m/sec? In m/min?

62. Use Table 7 to answer the following questions:

TABLE 7: Rates of Travel

Traveler	Speed in km/h
Common garden snail, slowest	0.006
Common garden snail, fastest	0.05
Giant tortoise, slowest	0.27
Human, walking	5
Human, bicycling	25
Hare	50
Cheetah	97
Jet airplane	1000
SST	2250

(a) By how many orders of magnitude is the fast snail faster than the slow snail?

(b) If a man bicycling could increase his speed by 2 orders of magnitude he would go as fast as which other entry in the table?

(c) By how many orders of magnitude faster is the hare than the the fastest snail?

63. Look up the area and the population of your town, and calculate the following:

(a) People per km^2

(b) km^2 per person

64. In 1971 the population of New York City was estimated at 7.90×10^6 persons. New York City has an area of 830 km^2. For the same year the population of Singapore was estimated at 2.15×10^6 persons. Singapore's area is 190 km^2. Which is more crowded, New York City or Singapore?

SECTION 8 POPULATION DENSITIES

The number of persons per unit area is a useful measure for determining how crowded a region is. We refer to this number as the population density. The population density, of course, differs for cities, rural areas, and deserts. When we divide the total population of a country by the total area of that country, we get the average population density for the entire country.

65. Using Table 1, which lists the areas of the continents, and Table 3, which lists the populations of the continents, calculate the average population density of each continent for 1970. Which continent was most crowded?

66. The following table lists the populations and the areas of selected countries.

Country	Population (persons)	Area (km^2)
Canada	2×10^7	1×10^7
U.S.A.	2×10^8	9×10^6
U.S.S.R.	3×10^8	2×10^7
Australia	1×10^7	8×10^6
India	6×10^8	3×10^6

(a) Which country has the largest population? Does this country also have the largest area?

(b) Which country is most crowded?

(c) Rank the countries from most crowded to least crowded.

67. Some parts of the world are more crowded than others. The following table shows population densities for various locations in the United States.

Place	Population Density (persons/km^2)
Wyoming	1.3
Washington, D.C.	4.8×10^3
Michigan	6.0×10^1
Alabama	2.6×10^1
New Jersey	3.7×10^2

(a) If the entire United States were populated as densely as Wyoming, what would the national population be?

(b) What would the U.S. population be if it were populated as densely as Washington, D.C.?

68. The most densely populated territory in the world is the Portuguese province of Macao off the southern coast of China. It has an estimated population of 321,000 (June 30, 1971) in an area of 16 km^2.

(a) What is the population density of Macao?

(b) What would be the population of the earth if it were as densely populated as Macao?

69. Population density is calculated as people/km^2. What would be the meaning of km^2/person?

70. Table 8 gives the area of cultivated land by continents. Compare this table with Table 1. What accounts for the difference?

TABLE 8: Cultivated Land by Continents, 1970

Continent	Area in km^2
Africa	2.04×10^6
Asia	4.49×10^6
Australia	4.7×10^5
Europe	4.53×10^6
North America	2.2×10^6
South America	1.2×10^6

71. Use Table 3 and Table 8 to answer the following questions.

(a) For each continent in 1970, how many people were fed by each km^2 of cultivated land?

(b) Which continent do you think is most likely to have a food shortage? Which is most likely to have a food surplus?

(c) What is the number of people/km^2 of cultivated land for the world taken as a whole?

(d) Which continents are above the world figure for people/km^2 of cultivated land? Which are below?

6. VARIABLES AND FUNCTIONS

SECTION 1 WHAT IS THE RULE?

Twelve different cards are to be divided in any way between two players. We allow for the possibility that one player will get them all. In how many different ways can this be done?

Form small groups and see if you can find all the possibilities. If you get tired before you have found the answer, try first to find the number of ways of dividing 2 cards. Then try 3 cards, then 4 cards. Do your answers appear to follow a general rule? Predict what the number of ways would be if you were given 5 cards. Verify your answer. Now what do you think the answer should be for 12 cards?

As you have seen, knowing a general rule can help us answer questions that might otherwise be very difficult. Let us consider another example.

If you have 5 T-shirts (white, blue, black, brown, and striped) and 3 pairs of jeans (white, brown, and blue), how many different outfits can you make from them? If you decided to wear the white jeans, there would be 5 possible choices of T-shirts, and so the number of ways in this case would be 5. Similarly, there

are 5 outfits with the brown jeans and another 5 with the blue. Adding up, we get 5 + 5 + 5 outfits. There are no other possible outfits. The answer is therefore $3 \times 5 = 15$ outfits.

We can generalize this reasoning to any number of jeans and T-shirts:

Number of outfits = (Number of jeans) × (Number of T-shirts)

We can simplify writing such rules by using symbols for the quantities involved. We can denote the number of jeans by j, the number of T-shirts by t, and the number of possible outfits by n. Then the rule becomes

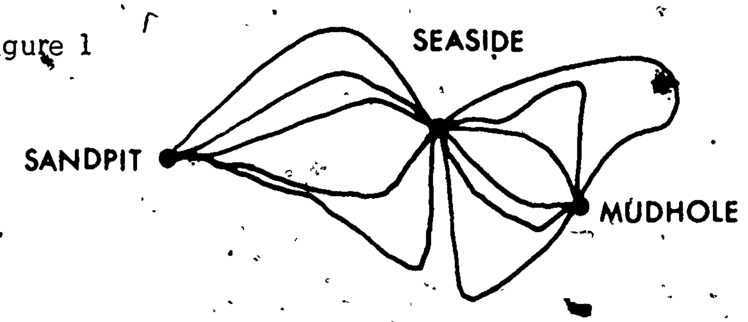
$$n = j \times t$$

?

1. How many outfits can you make from four different pairs of jeans and four different T-shirts?

2. The map in Figure 1 shows 5 roads from Sandpit to Seaside and 6 roads from Seaside to Mudhole. By how many possible routes can one travel from Sandpit to Mudhole? What rule did you use? State the rule with symbols.

Figure 1



3. By how many different routes is it possible to travel from Sandpit to Mudhole and then back to Seaside?

4. (a) George is now 15 years old and his sister Jane is 12. How old will Jane be when George's age is 20? 30? 55?

   (b) State a rule that expresses Jane's age in terms of George's age.

5. Suppose that you have a narrow strip of wood that is 1.50 m long and you wish to use all of it to make a rectangular picture frame.

   (a) What will be the height of the frame if you make the width 30 cm? 40 cm? 50 cm?

   (b) Give a general rule that expresses the height of the frame in terms of its width.

6. An entire school is to be taken on a picnic by bus. Each bus can hold 40 passengers. How many buses will be needed if the number of students and teachers is

   (a) 400?

   (b) 580?

   (c) 740?

   (d) 820?

   Describe in words how you would find the number of buses for any number of people.

SECTION 2 VARIABLES

Look again at the rule in the previous section:

Number of outfits = (Number of jeans) × (Number of T-shirts)

or

$$n = j \times t$$

If Margaret has 5 pairs of jeans and 7 T-shirts she can make 35

outfits. If someone collected 33 jeans and 29 T-shirts the rule tells us that $33 \times 29 = 957$ outfits could be made. The numbers of jeans, T-shirts, and outfits can vary from one case to another and are, therefore, called <u>variables</u>.

We can choose what the numbers of jeans and T-shirts will be. The corresponding number of outfits will then be given uniquely by the rule. For this reason, we call the numbers of jeans and T-shirts in this problem <u>independent variables</u>. The number of outfits, which depends on these variables according to the rule, is the <u>dependent variable</u>.

Another common way of saying the same thing is that the number of outfits is a <u>function</u> of the number of jeans and the number of T-shirts.

Here is an example of a different function. If you earned \$5.00 and spent some of it, the amount you would have left would be a function of how much you have spent. Specifically,

$$\text{Amount left} = 5.00 - \text{amount spent}$$

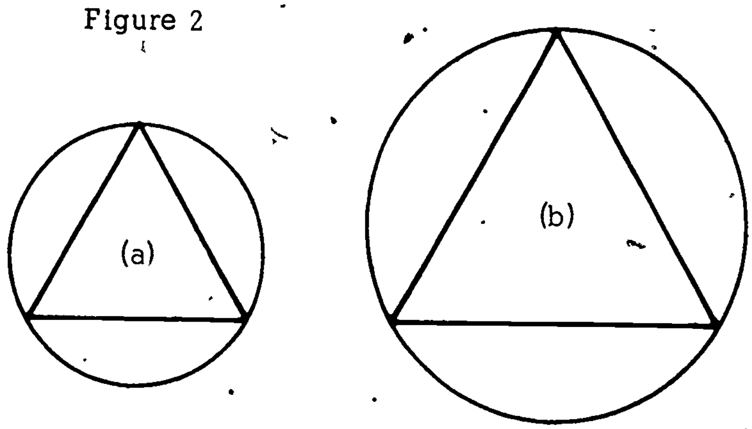
The amount spent is the independent variable and the amount left is the dependent variable.

We can write this rule using symbols. Let $\underline{s}$ be the amount of money spent (in dollars) and $\underline{\ell}$ the amount left (in dollars). The rule can now be written as

$$\ell = 5.00 - s$$

7. The diagram in Figure 2 (b) is similar to the diagram in Figure 2 (a) but is magnified by a scaling factor of 1.5. Let p denote the length of any line in (a) and q the length of the corresponding line in (b). Write the rule that expresses q as a function of p.

Figure 2



8. In Problem 6, dealing with the school picnic, which is the independent variable? Which is the dependent variable?

9. (a) Express the volume of a block as a function of its length, width, and height.

   (b) What are the independent variables that determine the area of the base of the block?

10. Express the area of the surface of a cube as a function of the length of its edge.

11. Suppose that in a supermarket you bought some beef at $1.70 per lb., coffee at $1.60 per lb., and breakfast cereal at 60 cents per package. Express the total bill as a function of the amounts of beef and coffee and the number of packages of cereal that you bought. Identify any symbols that you use.

12. Using symbols, express the perimeter of a square as a function of the length of its sides.

13. (a) Which are the independent and dependent variables in Problem 5?

(b) Express the area enclosed by the frame as a function of its width.

(c) What would this area be if the width were 30 cm? 40 cm? 50 cm?

14. Suppose that you set out from Sandpit (Figure 1) knowing that there were 5 roads to Seaside but not knowing how many roads there were from Seaside to Mudhole. Using symbols, state the rule that expresses the total number of possible routes from Sandpit to Mudhole as a function of the number of roads from Seaside to Mudhole. In this rule which are the independent and dependent variables?

SECTION 3 WAYS TO WRITE FUNCTIONS

There are some generally accepted ways of writing mathematical expressions that make them shorter and simpler.

We can write the product of two or more quantities represented by symbols by simply writing the symbols side by side without the multiplication sign, ×. For example, if $\underline{a}$ denotes the length of a block, $\underline{b}$ its width, and $\underline{c}$ its height, we may write the formula for its volume, $\underline{V}$, as

$$V = abc$$

We can use a similar convention if we wish to multiply some definite number by a quantity represented by a symbol. If hot dogs at a stand cost 40 cents each, then the cost of n hot dogs could be written as $40n$ cents, or $0.4n$ dollars.

In any product involving symbols and a numeral, the accepted notation is to write the numeral on the left. The cost in cents of n hot dogs is written as $40n$, not $n40$.

15. Could you omit the multiplication sign from the following? State reasons for your answer.

    (a) 7×3

    (b) $5 \times 6 \times 2$

    (c) 4×4

16. In simplified form, write:

    (a) $p \times q$

    (b) $p \times q \times r$

    (c) $2 \times a \times b$

    (d) $c \times 12$

    (e) $c \times 5 \times d$

17. Write the following in simplified form.

    (a) $3 \times a \times 4$

    (b) $p \times 2 \times q \times 3$

    (c) $5 \times 2.4 \times p$

    (d) $q \times 7.9$

    (e) 7×8

We often have to work with expressions that are sums or differences of quantities, some of which may be products of several factors. Expressions of this kind can be quite cumbersome to write. Consider, for example, an expression such as

$$(7a) + (bc) - (5c) + (2 \times 4)$$

To simplify the writing in cases like this, the following rules have been adopted.

(i) When the factors in any product are not all numbers, but include at least one symbol, the parentheses are omitted.

(ii) When all the factors in a product are numbers, parentheses are needed.

The preceding expression would therefore be written as

$$7a + bc - 5c + (2 \times 4)$$

Recalling what we learned at the beginning of this section, we see that the parentheses can be omitted precisely in those cases in which the multiplication sign can be omitted.

18. Write, as simply as possible:

(a) $(3 \times a \times b \times c) + (5 \times b \times c \times d) + (9 \times c) - (3 \times 5)$

(b) $(1.5 \times p \times q) - (a \times b \times c)$

(c) $(6 \times m) + (3 \times 2 \times n)$

(d) $(3.2 \times p \times q) + ((-2) \times 4 \times r) - (8 \times 6)$

19. A customer at a supermarket bought $\underline{p}$ packets of cookies at 60 cents per package, $\underline{d}$ cartons of eggs at 97 cents per carton, and $\underline{s}$ bottles of soda at 35 cents per bottle. As simply

as possible, express the total bill in cents as a function of $\underline{p}$, $\underline{d}$, and $\underline{s}$.

20. Suppose that a customer made the same purchases as in Problem 19 but was entitled to a refund of 10 cents for returned empty bottles. Again, as simply as you can, express the bill in cents as a function of $\underline{p}$, $\underline{d}$, and $\underline{s}$.

21. Express the surface area of a block as a function of its length, width, and height. Use symbols and write the expression in as simple a form as you can.

22. An Automobile Association of America (AAA) publication estimates on the basis of a number of tests that the average driver takes $\frac{3}{4}$ of a second before reacting to an observed danger. If $\underline{v}$ denotes the driver's speed in m/sec and $\underline{d}$ the distance in meters that will be traveled during the $\frac{3}{4}$-second time lag, express $\underline{d}$, as simply as possible, in terms of $\underline{v}$.

You learned in Chapter 3 to write 10^2 for 10×10, 10^3 for $10 \times 10 \times 10$, and so on. We can use this notation quite generally for repeated multiplication of any number. For instance, we can write 7^2 for 7×7 and 13^4 for $13 \times 13 \times 13 \times 13$.

The number 9^4, $9 \times 9 \times 9 \times 9$, is read as "9 to the 4th." In the same way $2.1 \times 2.1 \times 2.1 \times 2.1 \times 2.1 = (2.1)^5$ is read as "2.1 to the 5th."

We can also use this notation with symbols and write a^2 for $a \times a$, b^3 for $b \times b \times b$, etc. The product $a \times a \times a \times a = a^4$ is read as "a to the 4th," and so on.

We can now write many functions more simply. For example, if ℓ is the side of a square and A its area, we can write the area as a function of the length of the side:

$$A = \ell^2$$

Similarly, if s is the edge of a cube and V its volume, we may write

$$V = s^3$$

For these reasons the quantity ℓ^2 is often referred to as "ℓ squared" or "the square of ℓ," and s^3 is called "s cubed" or "the cube of s." There are no special names for other powers.

To avoid confusion it has been generally agreed that an expression such as pq^2 stands for $p(q^2)$ and not $(pq)^2$.

23. Write, as simply as possible:
    (a) $a \times a \times a \times b \times b$
    (b) $d \times d \times e$
    (c) $s \times s \times t \times t$
    (d) $k \times m \times k + k$
    (e) $c \times d + c \times c$

24. What is wrong with each of the following statements?
    (a) $a^2 + b = a \times a \times b$
    (b) $d \times d \times d = 3d$
    (c) $(2a)^2 = 2a^2$
    (d) $-c^2 = (-c) \times (-c)$

SECTION 4 FINDING VALUES OF FUNCTIONS

Suppose we are given a function, f, as defined by

$$f = 7a + bc - 5c + (2 \times 4)$$

and numerical values for the independent variables, $a = 1$, $b = 2$, and $c = 3$. To find the value of this function we replace each symbol in the expression by the corresponding numerical value:

$$f = (7 \times 1) + (2 \times 3) - (5 \times 3) + (2 \times 4)$$

Note that we used parentheses according to Rule (ii) in Section 3. Performing the indicated operations, gives us

$$f = 7 + 6 - 15 + 8$$
$$= 6$$

If we are given different numerical values for a, b, and c, the value of the function, f, may be different. For example, if $a = 4$, $b = 3$, and $c = (-1)$, then the value of the function is now given by

$$f = (7 \times 4) + (3 \times (-1)) - (5 \times (-1)) + (2 \times 4)$$
$$= 28 + (-3) - (-5) + 8$$
$$= 38$$

25. Evaluate $5x + 4y - 3xy$ in the following cases.

(a) $x = 4$, $y = 1$

(b) $x = 1$, $y = 0$

(c) $x = 0$, $y = 1$

(d) $x = 9$, $y = -2$

26. Evaluate $10a + 2abc - 9$ in the following cases.

(a) $a = 1$, $b = 2$, $c = 3$

(b) $a = -1$, $b = 2$, $c = 2$

(c) $a = 0.5$, $b = 4$, $c = 1$

27. When we are given a temperature on the centigrade scale and we wish to express it on the Fahrenheit scale, the rule is to multiply the centigrade temperature by 1.8 and add 32 to the product. Use symbols to express this rule as simply as you can. Be sure to define your symbols.

28. Evaluate the Fahrenheit temperatures that correspond to the following centigrade temperatures.

(a) $100^{\circ}C$

(b) $0^{\circ}C$

(c) $20^{\circ}C$

(d) $-40^{\circ}C$

(e) $37^{\circ}C$

29. Use the formula $A = \ell^2$ to find the area of a square with side $\ell = 8.5$ cm.

30. Use the formula $V = s^3$ to find the volume of a cube with edge $s = 5$ cm.

31. If the base of a prism is a square of side $\underline{s}$ and the height of the prism is $\underline{h}$, its volume, $\underline{V}$, is given by $V = s^2h$. Find $\underline{V}$ when

(a) $s = 4$ and $h = 6$

(b) $s = 6$ and $h = 4$

32. Evaluate $p^2 + 8$ when

(a) $p = 8$

(b) $p = 2.5$

(c) $p = 1$

(d) $p = (-4)$

33. Evaluate $a^2 - 3a + 2$ for the following values of $\underline{a}$.

(a) $a = 4$

(b) $a = 0$

(c) $a = 2$

(d) $a = (-1)$

34. Evaluate $4s^3$ for the following values of s.

(a) $s = 1$

(b) $s = (-1)$

(c) $s = 3$

(d) $s = 10$

35. Evaluate $d^2 + f^3$ when $d = 5$ and $f = (-2)$.

36. Which is greater?

(a) 2^3 or 3^2

(b) 4^3 or 3^4

(c) 2^4 or 4^2

37. The volume of a sphere is approximately equal to 52 percent of the volume of a particular cube. The edge of this cube is equal in length to the diameter of the sphere.

(a) Express the volume of the sphere as a function of its diameter.

(b) Find the volume of a sphere that has a diameter of 6.0 cm.

(c) Find the volume of a sphere that has a radius of 6.0 cm.

38. It is estimated that the world's population is increasing by a factor of 1.2 every decade. (A decade is a period of 10 years.) In 1970 the world population was 3.6×10^9. Express as a function of n what the expected population could be n decades after 1970, if the present trend continues.

Use this expression to predict, on the same assumption, the world's population in the year 2000.

SECTION 5 CIRCLES AND CYLINDERS

Circles of different radii have different areas; therefore, the area of a circle must be a function of its radius. In this section we will derive a rule that tells us how to calculate the area of a circle when we know its radius.

We will begin by rearranging a circle into a shape that has an area we can easily calculate. First, we divide a circle into eight equal pieces, as shown in Figure 3(a). Then we reassemble the pieces so that they form a new shape, as in Figure 3(b).

Figure 3

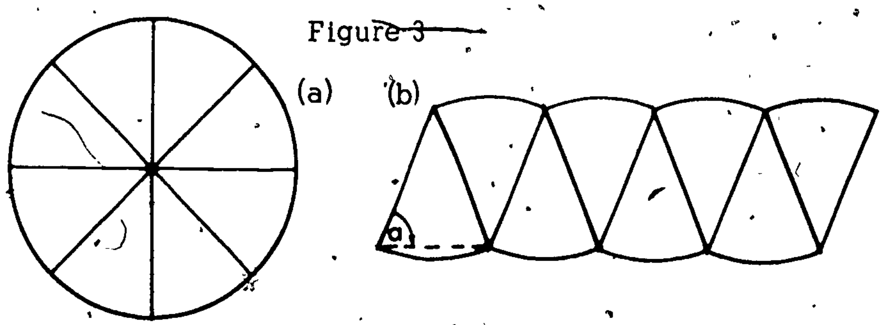


Figure 3(b) looks something like a parallelogram, except that the base and the top are not straight lines. Instead, the base

and the top each consist of four arcs of the circle. The area of the "parallelogram" is equal to that of the circle, because the new shape is made by rearranging the same eight pieces of the circle. Also, the four arcs that form the base are simply four of the eight equal arcs into which we had divided the circumference of the circle. Therefore, the combined length of these four arcs equals half of the circumference.

Let us divide the same circle into a greater number of equal pieces and reassemble them [see Figure 4(a) and (b)]. Figures 4 and 5 show the results of doing this for 16 pieces and for 32 pieces.

Figure 4

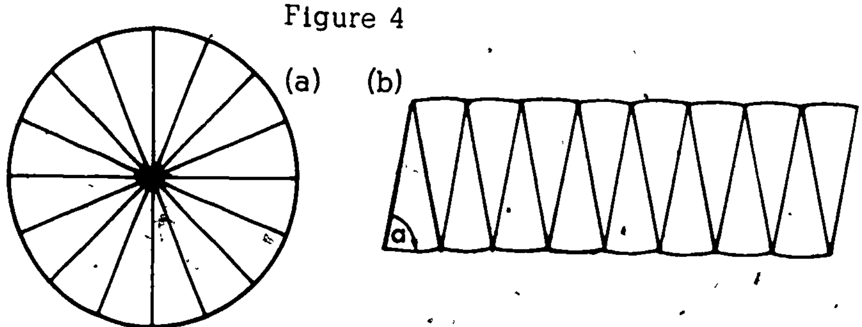
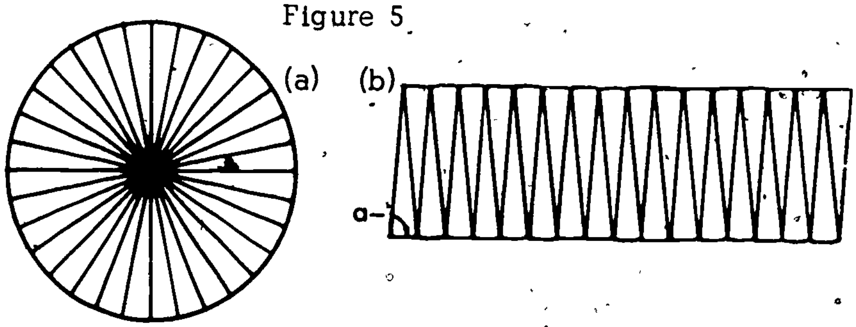


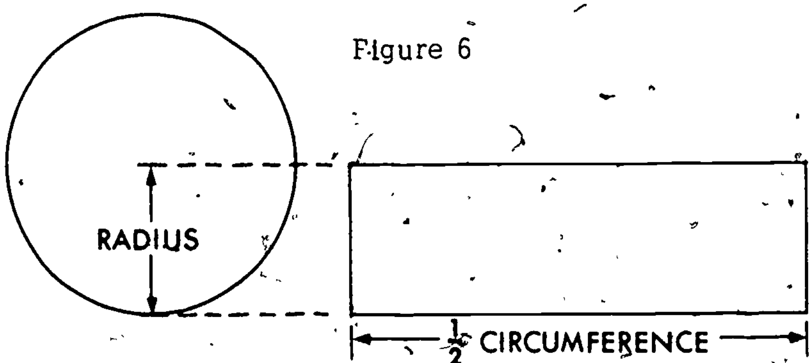
Figure 5



It does not matter into how many equal pieces we divide the circle: The area of the "parallelogram" that they form always equals the area of the circle. The sum of the lengths of the arcs that make

up the base remains equal to half the circumference of the circle. By comparing Figures 3, 4, and 5, we can see that as the number of pieces increases, the base of the "parallelogram" becomes more like a straight line. Now let us also look at the angle labeled a. As the number of pieces increases, this angle gets closer and closer to 90°. Therefore, with a large enough number of pieces the "parallelogram" will look just like a rectangle having a base equal to half the circumference and a height equal to the radius (Figure 6).

Figure 6



How can we express half the circumference of a circle as a function of its radius? Since circumference = $\pi \times$ diameter, $\frac{1}{2}$ circumference = $\pi \times$ radius, or simply πr. You will recall that the area of a rectangle is given by the product of its base and its height. Therefore, the area of the circle is given by the rule

$$A = (\pi r)r = \pi r^2$$

39. Calculate the area of a circle if its radius is
   (a) 12 cm
   (b) 4.4 cm
   (c) 15 cm

40. What is the area of the largest circle that can be cut from a square piece of leather 13 cm on a side?

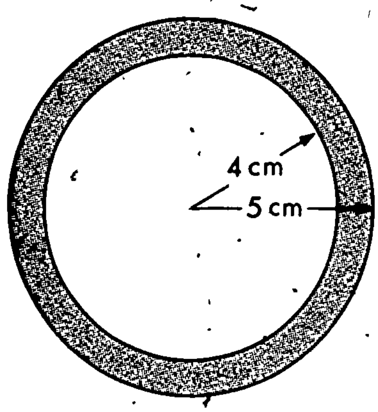
41. (a) Without calculating, decide which area is greater, that of one circle with a radius of 12 cm or that of two circles each 6 cm in radius.

(b) Check your answer by calculating these areas.

42. A pizza that has a 10-inch diameter costs $2 in a certain store. If you were the store manager and decided to sell 14-inch pizzas, about how much should you charge? Why?

43. In terms of π, find the area of the shaded ring shown in Figure 7. Could you draw a circle that has an area exactly equal to that of the ring?

Figure 7



44. (a) Consider a cylinder that has a height $\underline{h}$ and a circular base of radius $\underline{r}$. Express its volume $\underline{V}$ as a function of $\underline{h}$ and $\underline{r}$.

(b) What is the volume of a tin can that has a height of 15 cm and a radius of 6 cm?

45. If a can is 8.3 cm tall and has a <u>diameter</u> of 6.6 cm, what volume of soup could it hold?

46. A solid aluminum cylinder has a radius r and a height of 10 cm. Each cm^3 of aluminum weighs 2.8 g. Express the weight of the cylinder as a function of r.

47. The radii and heights of three cylinders A, B, and C are given in the following table. Which cylinder has the greatest volume and which has the least volume?

	Radius (in cm)	Height (in cm)
A	1	8
B	2	4
C	4	2

48. Express the total surface area of a circular cylinder as a function of its radius and height.

SECTION 6 PERMUTATIONS

We saw in Section 1 that if we have j jeans and t T-shirts we can make jt outfits. This rule is a special case of a more general rule. Suppose that we have to perform a pair of operations. If a denotes the number of ways of doing the first operation, b the number of ways of doing the second, and N the number of ways of doing the pair, then

$$N = ab$$

49. A customer in a restaurant is deciding among the following items, which he has selected from the menu.

Main Dish	Dessert
Fried chicken	Apple pie
Fried fish	Cheese cake
Seafood	Ice cream
Steak	

In how many different ways can he choose his meal?

50. A girl and a boy have to be chosen to represent their class at a meeting. If there are 14 girls and 12 boys in the class, in how many ways can the pair be selected?

51. You have to perform three operations and there are a ways of doing the first, b ways of doing the second, and c ways of doing the third. Express the number of ways, M, of doing all three as a function of a, b, and c.

52. Suppose that the restaurant customer of Problem 49 also wished to choose from among four possible beverages. What would now be the total number of possibilities open to him?

53. Suppose we have two objects, ● and ■, and we wish to arrange them in line. Clearly, there are two arrangements possible, either ●■ or ■●. In how many ways can three different objects be arranged in line? Explain your reasoning.

The different orders in which a given number of objects can be arranged on a line are called permutations. For two objects ● and ■, there are two permutations: ●■ and ■●. In Problem 53 you found the permutations of three objects. Clearly, the number of permutations depends on the number of objects. What is the rule that expresses this dependence?

Let n denote the number of objects. To arrange the objects we must decide which to put in the first place, which to put in the second, and so on. There are n possibilities for the first place. The number of possibilities left for the second place is $(n-1)$ because one of the objects has already been used to fill the first place. The number of ways of filling the first two places is therefore $n(n-1)$.

The number of possibilities for the third place is $(n-2)$, since that is the number of objects left now that the first two have been selected. The number of ways of filling the first three places is therefore $n(n-1)(n-2)$. We continue in this way until all the objects have been placed. The number of possibilities at each step is one less than at the previous step. For that last place there is only one object left.

If P denotes the number of permutations of n objects, we see that

$$P = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

that is, the product of all positive integers up to n. The product, $n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$, is usually written more compactly as $n!$, which is read as "n factorial." Therefore,

$$P = n!$$

54. Calculate the following.
    (a) 6!
    (b) 9!

(c) 10!

(d) 1!

55. In how many ways can six different objects be arranged?

56. There were seven candidates on the ballot in a presidential primary election. Assuming that there were no ties, state how many different outcomes were possible.

57. In a track meet five girls have entered the 1000-m race. In how many ways can the five girls be positioned in the five starting lanes?

58. How many 5-digit numbers can you make that contain all the odd digits 1, 3, 5, 7, 9 with each occurring once?

59. Seven competitors take part in a race. At the end of the race the names of who came in first, second, and third are announced. How many possible outcomes are there?

60. A group of 3 boys and 3 girls is to be photographed with the boys seated in a row and the girls standing behind them. In how many ways can the group be arranged?

61. How many even numbers having 5 digits can you make if the first digit on the left may not be 0?

62. How many different 4-digit numbers are there that have the first digit different from 0, the second odd, and the last 2 digits different from each other?

At the beginning of this chapter we discussed in how many ways we could divide a number of different cards between two players? We can now answer the question completely.

Suppose there were 5 cards in all. For each card we would have to decide whether to give it to the first player or the second. The number of possible decisions for each card is therefore 2. Since the same decision has to be made for each of the 5 cards, the total number of possible decisions is

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

It is also clear that if there were 12 cards, the number of possible decisions would be

$$2^{12} = 4096$$

Quite generally, if we denote the number of cards by n and the number of possible ways of dividing them between two players by D, then

$$D = 2^n$$

63. A set of c cards is to be divided among p players. Let N denote the number of ways in which the cards can be divided, and express N as a function of c and p.

64. Five different objects are to be divided among three people. In how many ways can this be done?

65. Before Mercedes went away to college, she left at home a shelf of 15 books and told her younger sister Amanda to take any of them that she wanted. How many possible choices of one or more books could Amanda make?